Investigations on the differential equation for the hypgeometric series*

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§. 1

It has been known since Euler that the definite integral

$$y = \int_0^1 V du,$$

with

$$V = u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha},$$

satisfies the differential equation

$$(1) \quad x(1-x)y'' + (\gamma - (\alpha+\beta+1)x)y' - \alpha\beta y = 0.$$

To demonstrate this, following Euler's procedure (*Institutiones calculi integralis, Vol. II, Sect. I, Chap. X, Problem 130*), one just has to introduce the integral $\int V du$ for y on the left–hand side of (1), to form y' and y'', i.e. $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, by differentiation under the integral sign, and finally simplify the resulting

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expression. Then, one will not obtain 0 on the right–hand side initially, but the expression[1]

$$-\alpha u^{\beta}(1-u)^{\gamma-\beta}(1-xu)^{-\alpha-1} = -\alpha \frac{u(1-u)}{1-xu} V.$$

Since the latter vanishes for $u = 0$ and $u = 1$, while assuming β and $\gamma - \beta$ to be positive, of course, the definite integral $y = \int\limits_0^1 V du$, in the case it is meaningful, will integrate (1).

It has been ignored in the past that before–mentioned expression also vanishes for $u = \pm\infty$, provided $\gamma - \alpha - 1$ is negative, such that, aside from the boundaries 0 and 1, the boundaries 0 and $-\infty$, 1 and ∞ exhibit integrals of (1) as well. Taking this into account one arrives at the following result:

"$y = \int\limits_g^h V du$ satisfies equation (1), if g and h denote two of the values 0, 1, $\pm\infty$ and we have

$$\left[\frac{u(1-u)}{1-xu} V\right]_g^h = 0;$$

provided the integral is meaningful, of course."

We use the familiar notation, according to which $[f(u)]_g^h$ represents the difference $f(h) - f(g)$.

Looking at the composition of the expression V, it was obvious to add the value $\frac{1}{x}$ to the values of the limits of the integral $\int V du$ just considered, the

[1]Following Euler, one might express this by saying that the differential quotient, taken with respect to u,

$$-\alpha \frac{d}{du}\left[\frac{u(1-u)}{1-xu} V\right],$$

can be represented in the form

$$A\frac{d^2V}{dx^2} + B\frac{dV}{dx} + CV,$$

where A, B, C are the quantities $x(1-x)$, $\gamma - (\alpha+\beta+1)x$, $-\alpha\beta$ independent from u that occur in equation (1).

first of which make the quantities u and $1-u$ equal to zero, since for $\frac{1}{x}$ the term $1-xu$ vanishes. By having substituted $y=\int_g^{\frac{\varepsilon}{x}} Vdu$ in the left–hand side of (1) at first, where ε denotes a constant, the following simplification yielded

$$-(\gamma-\beta-1)\varepsilon^{\beta}(1-\varepsilon)^{1-\alpha}x^{1-\gamma}(x-\varepsilon)^{\gamma-\beta-2}+\alpha g^{\beta}(1-g)^{\gamma-\beta}(1-xg)^{-\alpha-1}$$

and hence one was able conclude for $\varepsilon=1$ that

$$y=\int_g^{\frac{1}{x}} Vdu$$

satisfies equation (1) as well, if

$$\frac{u(1-u)}{1-xu}V$$

vanishes for $u=g$ and $1-\alpha$ is positive; here, we assume that the integral has a definite value.

Therefore, one has six definite integrals and, as it is easily demonstrated, as many *different* solutions to equation (1) (i.e. such no two of which have a constant coefficient). By assuming, as we will also do in the following, x to be positive, to which case the one for negative x is easily reduced, we present these solutions alongside the conditions for them to satisfy equation (1):

1)	if	β	and	$\gamma-\beta$	are positive,	$y=\int_0^1 V du,$
2)	″	β	″	$\alpha+1-\gamma$	″	$y=\int_0^{-\infty} V du,$
3)	″	$\gamma-\beta$	″	$\alpha+1-\gamma$	″	$y=\int_1^{\infty} V du,$
4)	″	β	″	$1-\alpha$	″	$y=\int_0^{\frac{1}{x}} V du,$
5)	″	$\alpha+1-\gamma$	″	$1-\alpha$	″	$y=\int_{\frac{1}{x}}^{\infty} V du,$
6)	″	$\gamma-\beta$	″	$1-\alpha$	″	$y=\int_1^{\frac{1}{x}} V du.$

To understand the meaning of these integrals more clearly, one might express them in terms of hypergeometric series. For, it is well–known that $\int_0^1 u^{\lambda}(1-u)^{\mu}(1-au)^{\nu}du$, aside from a constant factor, is equal to the hypergeometric series, which, following GAUSS, is denoted by $F(-\nu,\lambda+1,\lambda+\mu+2,a)$. Furthermore, it is easily seen that the limits of the six integrals can be transformed into 0 and 1 by a suitable transformation without the function under the integral losing its form $u^p(1-u)^q(1-bu)^r du$. I list the six solutions, expressed in terms of hypergeometric series, at which one arrives this way, together with the substitutions used in the process.

1) $F(\alpha, \beta, \gamma, x)$, Substitution $u = v$,

2) $x^{-\alpha} F\left(\alpha, \alpha+1-\gamma, \alpha+\beta+1-\gamma, \frac{x-1}{x}\right)$, " $u = \frac{v-1}{v}$,

3) $x^{-\alpha} F\left(\alpha, \alpha+1-\gamma, \alpha+1-\beta, \frac{1}{x}\right)$, " $u = \frac{1}{v}$,

4) $x^{-\beta} F\left(\beta, \beta+1-\gamma, \beta+1-\alpha, \frac{1}{x}\right)$, " $u = \frac{v}{x}$,

5) $x^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x)$, " $u = \frac{1}{xv}$,

6) $x^{\alpha-\gamma}(1-x)^{\gamma-\alpha-\beta} F\left(\gamma-\alpha, 1-\alpha, \gamma+1-\alpha-\beta, \frac{x-1}{x}\right)$, " $u = \frac{1}{x+(1-x)v}$,

To each of these solutions one finds three equal, only formally different, ones, if one, already having transformed the limits of the integrals to 0 and 1 by the above substitutions, additionally uses three new substitutions leaving the boundaries unchanged, namely:

$$u = 1-v; \quad u = \frac{v}{1-x+vx}; \quad u = \frac{1-v}{1-vx}.$$

Applying them, Vdu transforms into

$$(1-x)^{-\alpha} v^{\gamma-\beta-1}(1-v)^{\beta-1}\left(1-\frac{xv}{x-1}\right)^{-\alpha} dv,$$
$$(1-x)^{-\beta} v^{\beta-1}(1-v)^{\gamma-\beta-1}\left(1-\frac{xv}{x-1}\right)^{\alpha-\gamma} dv,$$
$$(1-x)^{\gamma-\alpha-\beta} v^{\gamma-\beta-1}(1-v)^{\beta}(1-vx)^{\alpha-\gamma} dv,$$

respectively; thus, from $F(\alpha, \beta, \gamma, x)$ they lead to

$$
\begin{aligned}
&(1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right), \\
&(1-x)^{-\beta} F\left(\gamma-\alpha, \beta, \gamma, \frac{x}{x-1}\right), \\
&(1-x)^{\gamma-\alpha-\beta} F\left(\gamma-\alpha, \gamma-\alpha, \gamma, x\right).
\end{aligned}
$$

If we now collect the integrals transformed into hypergeometric series, we obtain six classes, each of which contains four equivalent solutions:

Class I

$$
\begin{aligned}
&1) \quad F(\alpha, \beta, \gamma, x) \\
&2) \quad (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x) \\
&3) \quad (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right) \\
&4) \quad (1-x)^{-\beta} F\left(\beta, \gamma-\alpha, \gamma, \frac{x}{x-1}\right).
\end{aligned}
$$

Class II

$$
\begin{aligned}
&1) \quad x^{-\alpha} F\left(\alpha, \alpha+1-\gamma, \alpha+\beta+1-\gamma, \frac{x-1}{x}\right), \\
&2) \quad x^{-\beta} F\left(\beta, \beta+1-\gamma, \alpha+\beta+1-\gamma, \frac{x-1}{x}\right), \\
&3) \quad F\left(\alpha, \beta, \alpha+\beta+1, 1-x\right), \\
&4) \quad x^{-1-\gamma} F\left(\alpha+1-\gamma, \beta+1-\gamma, \alpha+\beta+1-\gamma, 1-x\right).
\end{aligned}
$$

CLASS III

1) $x^{-\alpha}\left(\alpha, \alpha+1-\gamma, \alpha+1-\beta, \dfrac{1}{x}\right)$

2) $x^{\beta-\gamma}(1-x)^{\gamma-\alpha-\beta}F\left(1-\beta, \gamma-\beta, \alpha+1-\beta, \dfrac{1}{x}\right),$

3) $(1-x)^{-\alpha}F\left(\alpha, \gamma-\beta, \alpha+1-\beta, \dfrac{1}{1-x}\right)$

4) $x^{1-\gamma}(1-x)^{\gamma-\alpha-1}\left(\alpha+1-\gamma, 1-\beta, \alpha+1-\beta, \dfrac{1}{1-x}\right).$

CLASS IV

1) $x^{-\beta}F\left(\beta, \beta+1-\gamma, \beta+1-\alpha, \dfrac{1}{x}\right),$

2) $x^{\alpha-\gamma}(1-x)^{\gamma-\alpha-\beta}F\left(1-\alpha, \gamma-\alpha, \beta+1-\alpha, \dfrac{1}{x}\right),$

3) $(1-x)^{-\beta}F\left(\beta, \gamma-\alpha, \beta+1-\alpha, \dfrac{1}{1-x}\right),$

4) $x^{1-\gamma}(1-x)^{\gamma-\beta-1}F\left(\beta+1-\gamma, 1-\alpha, \beta+1-\alpha, \dfrac{1}{1-x}\right).$

CLASS V

1) $x^{1-\gamma}F\left(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x\right),$

2) $x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}F\left(1-\alpha, 1-\beta, 2-\gamma, x\right),$

3) $x^{1-\gamma}(1-x)^{\gamma-\alpha-1}F\left(\alpha+1-\gamma, 1-\beta, 2-\gamma, \dfrac{x}{x-1}\right),$

4) $x^{1-\gamma}(1-x)^{\gamma-\beta-1}F\left(\beta+1-\gamma, 1-\alpha, 2-\gamma, \dfrac{x}{x-1}\right).$

CLASS VI

1) $x^{\alpha-\gamma}(1-x)^{\gamma-\alpha-\beta}F\left(\gamma-\alpha, 1-\alpha, \gamma+1-\alpha-\beta, \frac{x-1}{x}\right)$,

2) $x^{\beta-\gamma}(1-x)^{\gamma-\alpha-\beta}F\left(\gamma-\beta, 1-\beta, \gamma+1-\alpha-\beta, \frac{x-1}{x}\right)$,

3) $(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma+1-\alpha-\beta, 1-x)$,

4) $x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\alpha, 1-\beta, \gamma+1-\alpha-\beta, 1-x)$.

These 24 series are the same that KUMMER listed in §. 8 of his article on the hypergeometric series in volume 15 of CRELLE's journal, about the meaning of which one will find more detailed explanations in the before–mentioned article. The investigation at hand hence presents the new result that the definite integrals, which are equal to those series, are all obtained from integration of the same expression between two of of the limits 0, 1, $\pm\infty$, $\frac{1}{x}$.

§. 2

Another entirely different kind of relations between the integrals of equation (1) is obtained by generalizing the investigations that GAUSS carries out in his work on mechanical quadratures. There (Para. 8) a function T of degree $n+1$ occurs, whose connection with $\int\limits_0^1 \frac{Tdt}{t-a}$ offered an opportunity for discovery of the following theorem:

"If $y = f(x)$ is an integral of differential equation (1), then

$$(2) \quad z = \int\limits_g^h \frac{t^{\gamma-1}(1-t)^{\alpha+\beta-\gamma}}{(t-x)^{\rho}} f(t)dt = \int\limits^h W f(t)dt$$

will be an integral of the differential equation

$$(3) \quad x(1-x)z'' + (\rho+1-\gamma-(2\rho+1-\alpha-\beta)x)z' - (\rho-\alpha)(\rho-\beta)z = 0,$$

provided that g as well as h has one of the values 0, 1, $\pm\infty$, and

$$(2^a) \quad \left[\frac{t^\gamma(1-t)^{\alpha+\beta+1-\gamma}}{(t-x)^\rho}\left(f'(t)+\rho\frac{f(t)}{t-x}\right)\right]_g^h = 0.$$

One might even put $h = x$; but in this case, the expression in parentheses has to vanish for $t = g$ and $1-\rho$ must be positive."

In order to prove this theorem, one may assume that $f(t)$ satisfies the differential equation

$$t(1-t)f''(t) + (\gamma - (\alpha+\beta+1)t)f'(t) = \alpha\beta f(t),$$

which, if multiplied by $t^{\gamma-1}(1-t)^{\alpha+\beta-\gamma}$, has the form

$$\alpha\beta t^{\gamma-1}(1-t)^{\alpha+\beta-\gamma}f(t) = \frac{d(t^\gamma(1-t)^{\alpha+\beta+1-\gamma})f'(t)}{dt}.$$

The latter, substituted for the integral on the right–hand side of equation (2), after an integration by parts, yields

$$\alpha\beta z = \left[\frac{t^\gamma(1-t)^{\alpha+\beta+1-\gamma}f'(t)}{(t-x)^\rho}\right]_g^h + \rho\int_g^h \frac{t^\gamma(1-t)^{\alpha+\beta+1-\gamma}f'(t)}{(t-x)^{\rho+1}}dt$$

and, after another integration by parts,

$$\alpha\beta z = \left[t(1-t)W\left(f'(t)+\frac{\rho f(t)}{t-x}\right)\right]_g^h - \rho\int_g^h \frac{d}{dt}\left(\frac{t(1-t)}{t-x}W\right)f(t)dt.$$

Now applying the transformation presented in the remark of §. 1, according to which, if one sets $u = \frac{1}{t}$, $u^2V = W$, we have

$$-\rho\frac{d}{dt}\left(\frac{t(1-t)}{t-x}W\right)$$
$$= x(1-x)\frac{d^2W}{dx^2} + (\rho+1-\gamma-(2\rho+1-\alpha-\beta)x)\frac{dW}{dx} - \rho(\rho-\alpha-\beta)W,$$

one arrives at the theorem formulated above.

The result contained in the latter can be cast into another form by comparison of both solutions $F(\alpha,\beta,\gamma,x)$ and $x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\alpha,1-\beta,2-\gamma,x)$

to equation (1), which are contained in expression 1) of class I and expression 2) of class V. For, since $F(1-\alpha, 1-\beta, 2-\gamma, x)$ is a solution ζ of

$$(1^a) \quad x(1-x)\zeta'' + (2-\gamma-(3-\alpha-\beta)x)\zeta' - (1-\alpha)(1-\beta)\zeta = 0,$$

the result of the before–mentioned comparison can be formulated in such a way that a solution ζ to (1^a), if multiplied by $x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}$, leads to a solution to (1). If one lets equation (3) take the place of (1^a) by increasing α, β, γ by $1-\rho$, by means of the above theorem it immediately follows that

$$(4) \quad Z = x^{\rho-\gamma}(1-x)^{\rho+\gamma-\alpha-\beta-1}\int_g^h \frac{t^{\gamma-1}(1-t)^{\alpha+\beta-\gamma}}{(t-x)^\rho} f(t)dt$$

becomes a solution to that equation, into which (1) is transformed at the same time, i.e. to

$$(5) \quad x(1-x)Z'' + (\gamma+1-\rho-(\alpha+\beta+3-2\rho)x)Z' + (\alpha+1-\rho)(\beta+1-\rho)Z = 0.$$

Specializing $\rho = 1$ here, (5) becomes identical to (1), and from an integral $f(x)$ of equation (1) one obtains the additional one

$$(6) \quad x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}\int_g^h \frac{t^{\gamma-1}(1-t)^{\alpha+\beta-\gamma}}{t-x} f(t)dt.$$

§. 3

The last formula provides us with an especially interesting result, if $f(t)$ is a finite series, in other words, a hypergeometric series the first or second element of which is a negative integer number $-n$. In the following paragraphs, up to §. 6 inclusively, the properties of those series will be the main topic of discussion.

Differentiating equation (1)

$$x(1-x)y'' + (\gamma-(\alpha+\beta+1)x)y' - \alpha\beta y = 0$$

with respect to x several times, one obtains

$$
\begin{aligned}
&x(1-x)y''' &&+ (\gamma+1-(\alpha+\beta+3)x)y'' &&- (\alpha+1)(\beta+1)y' &&= 0,\\
&x(1-x)y'''' &&+ (\gamma+2-(\alpha+\beta+5)x)y''' &&- (\alpha+2)(\beta+2)y'' &&= 0,\\
&\dots && \dots && \dots && .
\end{aligned}
$$

The result found by $(n-1)$–times differentiation, by multiplication by

$$x^{\gamma+n-2}(1-x)^{\alpha+\beta-\gamma+n-1},$$

is cast into this form:

$$\frac{d\{x^n(1-x)^n My^{(n)}\}}{dx} = (\alpha+n-1)(\beta+n-1)x^{n-1}(1-x)^{n-1}My^{(n-1)},$$

with

$$M = x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma}.$$

Differentiating this equation $(n-1)$ times once more, one obtains

$$\frac{d^n\{x^n(1-x)^n My^{(n)}\}}{dx^n} = (\alpha+n-1)(\beta+n-1)\frac{d^{n-1}\{x^{n-1}(1-x)^{n-1}My^{(n-1)}\}}{dx^{n-1}}$$

and hence by iterated application for any positive n this yields the equation

$$\frac{d^n\{x^n(1-x)^n My^{(n)}\}}{dx^n} = \alpha(\alpha+1)\cdots(\alpha+n-1)\cdot\beta(\beta+1)\cdots(\beta+n-1)My,$$

in which M denotes the same value as above.

If now y is a hypergeometric series of x terminating at the n–th power, i.e. if one sets $\beta = -n$, while α and γ remain arbitrary, one will have

$$y = F(-n,\alpha,\gamma,x)$$

and, by the previous equation,

$$F(-n,\alpha,\gamma,x) = \frac{x^{1-\gamma}(1-x)^{\gamma+n-\alpha}}{\gamma(\gamma+1)\cdots(\gamma+n-1)}\frac{d^n\{x^{\gamma+n-1}(1-x)^{\alpha-\gamma}\}}{dx^n},$$

or, setting $\alpha + n$ instead of α,

$$(7) \quad F(-n, \alpha+n, \gamma, x) = \frac{x^{1-\gamma}(1-x)^{\gamma-\alpha}}{\gamma(\gamma+1)\cdots(\gamma+n-1)} \frac{d^n\{x^{\gamma+n-1}(1-x)^{\alpha+n-\gamma}\}}{dx^n}.$$

On the one hand this expression shows that each finite hypergeometric series can be transformed into the elegant form of the right–hand side of (7), on the other hand it transforms the frequently occurring differential expression of the right–hand side into an expanded form of a product of powers into a simple hypergeometric series. For $\alpha = \gamma = 1$ one obtains

$$\frac{1}{1 \cdot 2 \cdots n} \frac{d^n\{x^n(1-x)^n\}}{dx^n} = F(-n, n+1, 1, x),$$

and for $x = \frac{1-\xi}{2}$

$$\frac{1}{2^n \cdot 1 \cdot 2 \cdots n} \frac{d^n(\xi^2-1)^n}{d\xi^n} = F\left(-n, n+1, 1, \frac{1-\xi}{2}\right),$$

i.e. on the left–hand side one finds the well–known function arising from the expansion of $\frac{1}{\sqrt{1-2h\xi+h^2}}$ according to powers of h. The expression of the latter on the right–hand side as a series was given by Dirichlet (*Crelle's Journal, Vol. 17, p. 39*). In similar manner, one will obtain an expansion of the n–th differential quotient $\frac{d^n(1-\xi^2)^{n-\frac{1}{2}}}{d\xi^n}$, which is known to be connected to $\cos n\varphi$ for the choice $\xi = \cos\varphi$.

§. 4

Without any difficulty, one will find the generating function of the expressions given by (7) of the same kind, as it was done in *Crelle's Journal, Vol. 2, p. 224* (p. 22 and 23 of this volume) in the before–mentioned case $\alpha = \gamma = 1$ in §. 3. One might even invoke Lagrange's formula, according to which

$$\chi(y)\frac{dy}{dx} = \chi(x) + \frac{h}{1}\frac{d[f(x)\chi(x)]}{dx} + \frac{h^2}{1 \cdot 2}\frac{d^2[f^2(x)\chi(x)]}{dx^2} + \cdots,$$

if, given x and y, the equation,

$$y - x = hf(y)$$

holds; then, one only needs to set

$$f(x) = x(1-x), \quad \chi(x) = x^{\gamma-1}(1-x)^{\alpha-\gamma}.$$

Letting

$$F(-n, \alpha+n, \gamma, x) = X_n$$

and, as above, $2x = 1 - \xi$, one obtains

$$\frac{x^{1-\gamma}(1-x)^{\gamma-\alpha}\{h-1+\sqrt{1-2h\xi+h^2}\}^{\gamma-1}\{h+1-\sqrt{1-2h\xi+h^2}\}^{\alpha-\gamma}}{(2h)^{\alpha-1}\sqrt{1-2h\xi+h^2}}$$
$$= \sum_{n=0}^{\infty} \frac{\gamma(\gamma+1)\cdots(\gamma+n-1)}{1\cdot 2\cdots n} h^n X_n.$$

This formula, which did not seem to be recommendable because of its simplicity, has not been investigated any further, aside from a certain special of it.

§. 5

By means of the binomial theorem, expand

$$(1+2h\xi+h^2)^{-c}$$

according to rising powers of h. Equating the series resulting this way to

$$\sum_{n=0}^{\infty} h^n Y_n,$$

it will be

$$Y_n = \frac{2c(2c+1)\cdots(2c+n-1)}{1\cdot 2\cdots n} F\left(-n, 2c+n, \frac{2c+1}{2}, x\right),$$

provided x and ξ are connected as in the previous paragraph, i.e., additionally

$$Y_n = 4^n \frac{c(c+1)\cdots(c+n-1)}{(2c+n)(2c+n+1)\cdots(2c+2n-1)} \frac{[x(1-x)]^{\frac{1}{2}(1-2c)}}{\Pi(n)} \frac{d^n[x(1-x)]^{\frac{1}{2}(2c+2n-1)}}{dx^n}.$$

§. 6

By the expressions denoted by X_n in §. 4, provided that γ and $\alpha + 1 - \gamma$ are positive, any function $\varphi(x)$ can only be expanded in one unique way such that assuming $\varphi(x) = \sum_{n=0}^{\infty} a_n X_n$ the constants are completely determined. For the purpose of demonstrating this theorem, one just has to show that

$$J_{m,n} = \int_0^1 X_m X_n x^{\gamma-1}(1-x)^{\alpha-\gamma} dx$$

vanishes, if the numbers m and n are different from each other. But X_n satisfies the differential equation

$$x(1-x)X_n'' + (\gamma - (\alpha+1)x)X_n' = -n(n+\alpha)X_n$$

such that

$$\begin{aligned} -n(n+\alpha)J_{m,n} &= \int_0^1 X_m \frac{d\{x^\gamma(1-x)^{\alpha+1-\gamma}X_n'\}}{dx} dx \\ &= \int_0^1 X_n \frac{d\{x^\gamma(1-x)^{\alpha+1-\gamma}X_m'\}}{dx} dx, \end{aligned}$$

i.e. becomes equal to $-m(m+\alpha)J_{m,n}$, from which one concludes that $J_{m,n}$ vanishes. For $m = n$ the value of the constant is easily calculated, since

$$n(n+\alpha)J_{n,n} = \int_0^1 X_n' X_n' x^\gamma (1-x)^{\alpha+1-\gamma} dx,$$

furthermore,

$$(n-1)(n+\alpha+1)\int_0^1 X_n' X_n' x^\gamma(1-x)^{\alpha+1-\gamma} dx = \int_0^1 X_n'' X_n'' x^{\gamma+1}(1-x)^{\alpha+2-\gamma} dx$$

etc. such that for $J_{n,n}$ one obtains the value

$$\frac{1}{\alpha+2n} \frac{\Pi(n)[\Pi(\gamma-1)]^2 \Pi(\alpha+n-\gamma)}{\Pi(\alpha+n-1)\Pi(\gamma+n-1)}.$$

§. 7

For a second integral of the differential equation, the first of which is X_n, by means of formula (6) of §. 2 one obtains the value

$$x^{1-\gamma}(1-x)^{\gamma-\alpha}\int_g^h \frac{t^{\gamma-1}(1-t)^{\alpha-\gamma}}{t-x}F(-n,\alpha+n,\gamma,t)dt,$$

which for $\alpha=\gamma=1$ becomes the one mentioned at the beginning of §. 2, if one writes $n+1$ instead of n and $g=0$, $h=1$.

According to §. 3, the above value can also be replaced with

$$x^{1-\gamma}(1-x)^{\gamma-\alpha}\int_g^h \frac{d^n\{t^{\gamma+n-1}(1-t)^{\alpha+n-\gamma}\}}{dt^n}\frac{dt}{t-x},$$

thus, if the values γ, α permit an integration by parts, also with

$$(8) \quad Z_n = x^{1-\gamma}(1-x)^{\gamma-\alpha}\int_g^h \frac{t^{\gamma+n-1}(1-t)^{\alpha+n-\gamma}}{(t-x)^{n+1}}dt.$$

The differential equation will then be completely integrated by

$$aX_n + bZ_n,$$

a and b denoting arbitrary constants.

§. 8

The results Gauss obtained by comparison of T and $\int_0^1 \frac{Tdt}{t-a}$ for the continued fraction expansion of the logarithmic series, can be transferred to the particular hypergeometric series $F(\alpha,1,\gamma,x)$ by comparison of X_n and Z_n. This way, almost without any calculation, the results on approximate fractions are obtained, which were initially found by resolution of linear equations (*Crelle's Journal, Vol., p. 208* and *Vol. 34, p. 297.*)

Let $\alpha + 1 - \gamma$ and γ, moreover, let $x > 1$; denoting the value of X_n for $x = t$ by T_n and setting

$$-W_n = \int_0^1 t^{\gamma-1}(1-t)^{\alpha-\gamma} \frac{T_n - X_n}{t-x} dt,$$

such that W_n is a entire function of degree $(n-1)$ of x, one immediately has the equation

$$X_n \int_0^1 \frac{t^{\gamma-1}(1-t)^{\alpha-\gamma}}{t-x} dt = W_n + \int_0^1 \frac{t^{\gamma-1}(1-t)^{\alpha-\gamma}}{t-x} T_n dt$$

and hence, if a and b denote easily computable constants,

$$\frac{a}{x} X_n F\left(\gamma, 1, \alpha+1, \frac{1}{x}\right) = W_n + b \int_0^1 \frac{t^{\gamma+n-1}(1-t)^{\alpha+n-\gamma}}{(t-x)^{n+1}} dt.$$

The integral multiplied by b, if expanded according to decreasing powers of x, starts with x^{-n-1} (the degree is $-(n+1)$); therefore, we have a function X_n of degree n, which multiplied by $F\left(\gamma, 1, \alpha+1, \frac{1}{x}\right)$ yields the entire function xW_n and a remainder of degree $(-n)$. It is well–known since Gauss's work on mechanical quadratures how this property of the X_n enables to immediately state the denominators of the continued fraction for $F\left(\alpha, 1, \gamma, \frac{1}{x}\right)$ in the way it results from Gauß's article on the hypergeometric series (Parag. 13), with the notation used there, the continued fraction being

$$\cfrac{x}{x - \cfrac{a}{1 - \cfrac{b}{x - \cfrac{c}{1 - \text{etc.}}}}}.$$

For, the denominator Q_{2n} of the $(2n)$–th approximate value has the form

$$Q_{2n} = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n,$$

the one of the $(2n+1)$–th

$$Q_{2n+1} = x(x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n),$$

if we count x as the first and $x - a$ as the second. Furthermore, Q_{2n} or Q_{2n+1}, if multiplied by $F\left(\alpha, 1, \gamma, \frac{1}{x}\right)$, has to be equal to an entire function of x, increased by a remainder of degree $-n$. Thus, Q_{2n} and Q_{2n+1} can only differ by the constant factors of $F(-n, \gamma + n - 1, \alpha, x)$ and $xF(-n, \gamma + n, \alpha + 1, x)$; having determined these factors properly, i.e. in such a way that the highest power of x get one as factor, this results in

$$
\begin{aligned}
Q_{2n} &= \quad x^n F\left(-n, 1 - \alpha - n, 2 - \gamma - 2n, \frac{1}{x}\right) \\
Q_{2n+1} &= x^{n+1} F\left(-n, -\alpha - n, 1 - \gamma - 2n, \frac{1}{x}\right).
\end{aligned}
$$

If γ and $\alpha + 1 - \gamma$ have different signs, this can obviously not alter the results.

§. 9

We now pass on to our last investigation, namely, answering the question, whether it is possible for any value of the elements to integrate differential equation (1) completely by simple definite integrals. Without any difficulty we see that the six definite integrals of §. 1 are not all valid simultaneously. Not only setting

$$V = u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha},$$

as previously, but also

$$W = u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-xu)^{-\beta},$$

the following table lists the cases in which $\int V du$ or $\int W du$ yield a solution to (1). The list is constructed with respect to the sings of α, β, $\gamma - \alpha$, $\gamma - \beta$ and, in order to reduce the number of cases, $\beta - \alpha$ is assumed to be non–negative.

	β	$\gamma-\beta$	α	$\gamma-\alpha$	$x>1$		$x<1$	
1.	$+$	$+$	$+$	$+$			$\int_0^1 Vdu$	
2.	$+$	$+$	$-$	$+$	$\int_{\frac{1}{x}}^1 Vdu$	$\int_0^{\frac{1}{x}} Vdu$	$\int_0^1 Vdu$	$\int_1^{\frac{1}{x}} Vdu$
3.	$+$	$-$	$+$	$+$	$\int_0^{-\infty} Wdu$	$\int_1^{\infty} Wdu$	$\int_0^1 Wdu$	$\int_0^{-\infty} Wdu$
4.	$+$	$-$	$+$	$-$	$\int_0^{-\infty} Vdu$		$\int_0^{-\infty} Vdu$	
5.	$+$	$-$	$-$	$+$	$\int_0^{\frac{1}{x}} Vdu$			
6.	$+$	$-$	$-$	$-$	$\int_0^{-\infty} Vdu$	$\int_0^{\frac{1}{x}} Vdu$	$\int_{\frac{1}{x}}^{\infty} Vdu$	$\int_0^{-\infty} Vdu$
7.	$-$	$+$	$-$	$+$	$\int_{\frac{1}{x}}^1 Vdu$		$\int_1^{\frac{1}{x}} Vdu$	
8.	$-$	$-$	$-$	$+$	$\int_{\frac{1}{x}}^1 Wdu$	$\int_1^{\infty} Wdu$	$\int_0^{\frac{1}{x}} Wdu$	$\int_{\frac{1}{x}}^{\infty} Wdu$
9.	$-$	$-$	$-$	$-$			$\int_{\frac{1}{x}}^{\infty} Vdu$	

Since for the case $x > 1$ as well as the case $x < 1$ one has to assign two different solutions, consulting the table it is quickly understood, when additional solutions have to be found.

§. 10

In order to find such solutions, one can invoke the theorem formulated in §. 2. For, this theorem gives the relations between the differential equations of two hypergeometric series, the elements of which are α, β, γ and $\rho-\alpha$, $\rho-\beta$, $\rho+1-\gamma$, such that $\rho-\alpha$, $\rho-\beta$, $\rho+1-\gamma$ go over into

$$\rho-(\rho-\alpha)=\alpha, \quad \rho-(\rho-\beta)=\beta, \quad \rho+1-(\rho+1-\gamma)=\gamma,$$

respectively. Now taking ρ in such a way that $\rho-\alpha$ is equal to a negative number $-n$, an integral of the first differential equation is the finite series

$$F(-n, \alpha-\beta-n, \alpha+1-\gamma-n, x) = f(x),$$

the second differential equation then is differential equation (1) itself; according to (2), one hence finds an integral of (1) as the formula

$$z = \int_g^h \frac{t^{\alpha-\gamma-n}(1-t)^{\gamma-\beta-n-1}}{(t-x)^{\alpha-n}} f(t)dt,$$

or, applying the transformation given in §. 3, the formula

$$(9) \quad Z = \int_g^h \frac{d^n\{t^{\alpha-\gamma}(1-t)^{\gamma-\beta-1}\}}{dt^n}(t-x)^{n-\alpha}dt,$$

provided that for constant g and h

$$\left[\frac{t^{\alpha+1-\gamma-n}(1-t)^{\gamma-\beta-n}}{(t-x)^{\alpha-n}}\left(f'(t)+\frac{\alpha-n}{t-x}f(x)\right)\right]_g^h = 0$$

and that for $h = x$ the expression in parentheses vanishes for $t = g$ and $n+1-\alpha$ is positive.

This result is also easily verified, and other similar statements are found with same ease, considering that after integration by parts

$$t^{\alpha-\gamma}(1-t)^{\gamma-\beta-1}(t-x)^{-\alpha}dt$$

remains under the integral on the right–hand side, if this operation is admissible, of course. But this expression, if integrated between g and h, is a solution of (1); if, e.g., g and h are equal to 0 and 1, respectively, the integration gives a solution of the third class. Hence one concludes immediately that, even if integration by parts is actually not allowed, Z is a solution to (1), if only the integral has a finite value.

§. 11

We can now complete our table from §. 9.

1) In the first case, there are only two integrals missing, if $x > 1$; Obviously, the following ones can be added:

$$\int_x^\infty \frac{d^n\{t^{\beta-\gamma}(1-t)^{\gamma-\alpha-1}\}}{dt^n}(t-x)^{n-\beta}dt,$$

$$\int_x^\infty \frac{d^n\{t^{\alpha-\gamma}(1-t)^{\gamma-\beta-1}\}}{dt^n}(t-x)^{n-\alpha}dt,$$

if n is assumed so large that $n+1-\beta$ or $n+1-\alpha$ are positive, respectively. If one is allowed to take $n = 0$, these integrals are of the III. and IV. class, respectively.

If $x < 1$, one integral is missing, which is assumed to be equal to $(1-x)^{-\beta}\zeta$, with ζ satisfying differential equation (1), if in it we change α, β, γ to β, $\gamma-\alpha$, $\beta+1-\alpha$, $\frac{1}{1-x}$. Confer form $3)$ of class IV. Hence it follows that

$$(1-x)^{-\beta}\int_{\frac{1}{1-x}}^\infty \frac{d^n\{t^{\gamma-\beta-1}(1-t)^{-\alpha}\}}{dt^n}\left(t-\frac{1}{1-x}\right)^{\alpha+n-\gamma}dt$$

can be considered as the missing integral, if $\alpha+n+1-\gamma$ is positive. For $n = 0$ an integral of class II. is obtained.

2) In the fourth case, if $x > 1$, one can obviously take

$$\int_x^\infty \frac{d^n\{t^{\beta-\gamma}(1-t)^{\gamma-\alpha-1}\}}{dt^n}(t-x)^{n-\beta}dt,$$

as a solution, under the constraint that $n+1-\beta$ is positive.

If $x < 1$, taking into account of form $1)$ of class II, let the missing integral be equal to $x^{-\alpha}\zeta$, where ζ satisfies the differential equation, into which (1) is transformed, if for α, β, γ, x one writes α, $\alpha+1-\gamma$, $\alpha+\beta+1-\gamma$, $\frac{x-1}{x}$, respectively. Hence one obtains

$$x^{-\alpha}\int\limits_{\frac{x-1}{x}}^{-\infty}\frac{d^n\{t^{\gamma-\beta-1}(1-t)^{\beta-1}\}}{dt^n}\left(t-\frac{x-1}{x}\right)^{n-\alpha}dt$$

as solution, under the condition that $n+1-\alpha$ is positive. For $n=0$ one obtains integrals from class III and IV.

3) In the fifth case one has to find an integral, if $x>1$. Considering the fourth integral of class I, put $z=(1-x)^{-\beta}\zeta$ and, as above, write $\gamma-\alpha$, β, γ, $\frac{x}{x-1}$ instead of α, β, γ, x; this way one finds

$$(1-x)^{-\beta}\int\limits_{\frac{x}{x-1}}^{\infty}\frac{d^n\{t^{\beta-1}(1-t)^{\alpha-1}\}}{dt^n}\left(t-\frac{x}{x-1}\right)^{n-\beta}dt,$$

provided that $n+1-\beta$ is positive.

If $x<1$, one obtains two integrals

$$(1-x)^{-\beta}\int\limits_{\frac{x}{x-1}}^{\infty}\frac{d^n\{t^{\beta-\gamma}(1-t)^{\alpha-1}\}}{dt^n}\left(t-\frac{x}{x-1}\right)^{n-\beta}dt$$

$$x^{-\beta}\int\limits_{\frac{1}{x}}^{\infty}\frac{d^n\{t^{\alpha-\gamma}(1-t)^{-\alpha}\}}{dt^n}\left(t-\frac{1}{x}\right)^{\gamma+n-\beta+1}dt$$

under the condition that $n+1-\beta$ and $\gamma+n-\beta$ are positive, respectively. For $n=0$ the integrals turn into those of class VI and I.

4) In the seventh case, for $x>1$ one finds

$$x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}\int\limits_{x}^{\infty}\frac{d^n\{t^{\gamma-\beta-1}(1-t)^{\alpha-\gamma}\}}{dt^n}(t-x)^{n+\beta-1}dt$$

and for $x<1$

$$x^{\alpha-\gamma}(1-x)^{\gamma-\alpha-\beta}\int\limits_{\frac{x-1}{x}}^{-\infty}\frac{d^n\{t^{\beta-\gamma}(1-t)^{-\beta}\}}{dt^n}\left(t-\frac{x-1}{x}\right)^{n+\alpha-1}dt$$

under the constraint that $n+\beta$ and $n+\alpha$ are positive, respectively. For $n=0$, integrals of class IV and I arise.

5) In the ninth case, for $x>1$ one obtains

$$x^{1-\gamma}\int\limits_{x}^{\infty}\frac{d^n\{t^{\beta-1}(1-t)^{-\alpha}\}}{dt^n}\,(t-x)^{\gamma+n-\beta-1}\,dt$$

and

$$x^{1-\gamma}\int\limits_{x}^{\infty}\frac{d^n\{t^{\alpha-1}(1-t)^{-\beta}\}}{dt^n}\,(t-x)^{\gamma+n-\alpha-1}\,dt,$$

finally, for $x<1$,

$$x^{1-\gamma}(1-x)^{\gamma-\beta-1}\int\limits_{\frac{1}{1-x}}^{\infty}\frac{d^n\{t^{-\beta}(1-t)^{\gamma-\alpha-1}\}}{dt^n}\left(t-\frac{1}{1-x}\right)^{n+\alpha-1}dt.$$

The conditions are that $\gamma+n-\beta$, $\gamma+n-\alpha$ and $n+\alpha$ are positive, respectively; for $n=0$ integrals of class III, IV and II arise.